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TRANSVERSE VIBRATIONS OF ELASTICALLY CONNECTED DOUBLE-STRING COMPLEX SYSTEM, PART I: FREE VIBRATIONS

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A theoretical vibration analysis of an elastically connected double-string system is presented. The double-string system is the simplest model of a complex continuous system, which is composed of two one-dimensional elastic solids attached by a Winkler elastic layer. The free and forced transverse vibrations of this system are considered. The present paper develops the free vibration theory, and a companion paper analyzes the forced vibrations. The motion of the system considered is described [1] by a non-homogeneous set of two partial differential equations, which are solved by using classical mathematical methods. The solutions of the free vibrations are derived from the Bernoulli–Fourier method. The boundary-value and initial-value problems are solved. The natural frequencies and natural mode shapes of vibration are determined. The free vibrations of an elastically connected double-string system are realized by synchronous and asynchronous deflections.

1. INTRODUCTION

The vibration problems of one-dimensional and two-dimensional continuous systems are important from a theoretical as well as from a technical point of view. Many modern engineering structures often make use of one-dimensional continuous elements resistant to tension but not to bending (for example, strings, cables, ropes, chains, etc.) [2–8]. A string, being the simplest model of one-dimensional continuous system has been a subject of great scientific interest for a considerable time. This fact is confirmed by the number of references collected. Fundamental theory for string vibrations is discussed in a number of monographs by e.g. Bishop and Johnson [2], Den Hartog [3], Fryba [4], Kaliski [5], Nowacki [6, 7], etc. The different aspects of string dynamics are treated by numerous investigators and many recent studies are devoted to the vibration problems of strings [9–92]. Among these publications, the works concerning the vibrations of a string supported on an elastic foundation are especially interesting [7, 36, 45, 59, 60, 74, 82, 85, 88].

The present paper deals with the transverse vibration analysis of an elastically connected double-string system. This mechanical system is an example of a complex continuous system. Complex continuous systems have theoretical and practical importance and have a wide application in aeronautics, cosmonautics, civil and mechanical engineering [4, 6, 9, 85, 89–93]. A system of two parallel strings continuously coupled by a linear elastic element constitutes an interesting vibratory structural system. The physical model of this system is the simplest form of a complex continuous system which is composed of two one-dimensional solids. The theoretical foundations of transverse vibrations of a two-string

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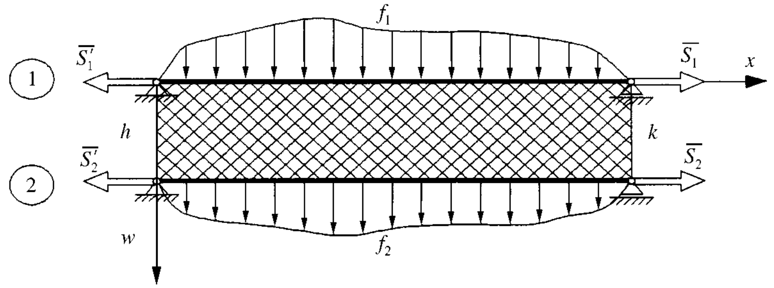

Figure 1. The physical model of an elastically connected double-string complex system.

system have been presented in the author's early works [9, 89–91]. The vibration problem of a similar elastically connected double-membrane system as an example of a complex continuous system consisting of two two-dimensional solids has been considered by the author in a recent paper [93].

In the present paper, the free transverse vibrations of two parallel strings continuously joined by a Winkler elastic element are analyzed and the complete exact theoretical solutions of free vibrations are formulated.

2. FORMULATION OF THE PROBLEM

The physical model of the vibrating system under consideration consists of two parallel, homogeneous and uniform strings attached together by a Winkler elastic element (see Figure 1). Both the strings have the same length and are supported at their ends. The strings are stretched under suitable constant tensions and subjected to arbitrarily distributed continuous loads. The small vibrations of the system with no damping are considered.

The transverse vibrations of a generally loaded double-string system are described by the following differential equations [9, 89–91]:

$$m_1\ddot{w}_1 - S_1 w_1'' + k(w_1 - w_2) = f_1, \qquad m_2\ddot{w}_2 - S_2 w_2'' + k(w_2 - w_1) = f_2, \tag{1}$$

where $w_i = w_i(x, t)$ is the transverse string deflection; $f_i = f_i(x, t)$ is the exciting distributed load; x, t are the spatial co-ordinate and the time; F_i is the cross-sectional area of the string; h is the height of an elastic element; k is the stiffness modulus of a Winkler elastic element; l is the string length; S_i is the tension of the string; ρ_i is the mass density; $m_i = \rho_i F_i$, $\dot{w}_i = \partial w_i/\partial t$, $w_i' = \partial w_i/\partial x$, $i = 1,2$.

The boundary and initial conditions for this problem have the form

$$w_i(0, t) = w_i(l, t) = 0, \tag{2}$$

$$w_i(x, 0) = w_{i0}(x), \qquad \dot{w}_i(x, 0) = v_{i0}(x), \qquad i = 1,2. \tag{3}$$

3. FREE VIBRATIONS

The governing equations for the free vibrations of an elastically connected two-string system are the following:

$$m_1\ddot{w}_1 - S_1 w_1'' + k(w_1 - w_2) = 0, \qquad m_2\ddot{w}_2 - S_2 w_2'' + k(w_2 - w_1) = 0. \tag{4}$$

These homogeneous partial differential equations are solved by the Bernoulli–Fourier method assuming the general solutions of equations (4) in the form

$$w_i(x, t) = X_i(x)T(t), \quad i = 1,2, \tag{5}$$

$$T(t) = A\sin(\omega t) + B\cos(\omega t), \tag{6}$$

where ω is the natural frequency of the system. By substituting expressions (5) into equations (4) one obtains a set of ordinary differential equations

$$S_1X_1'' + (m_1\omega^2 - k)X_1 + kX_2 = 0, \qquad S_2X_2'' + (m_2\omega^2 - k)X_2 + kX_1 = 0, \tag{7}$$

which have the following solutions:

$$X_1(x) = Ce^{\mathrm{i}rx}, \qquad X_2(x) = De^{\mathrm{i}rx}, \qquad \mathrm{i} = (-1)^{1/2}. \tag{8}$$

Substituting them into equations (7) results in the following system of homogeneous algebraic equations:

$$C[S_1r^2 - (m_1\omega^2 - k)] - Dk = 0, \qquad Ck - D[S_2r^2 - (m_2\omega^2 - k)] = 0, \tag{9}$$

for which there are non-trivial solutions when the cardinal determinant of the system coefficient matrix is equal to zero. This in turn yields the following characteristic equation:

$$r^4 - [(m_1\omega^2 - k)S_1^{-1} + (m_2\omega^2 - k)S_2^{-1}]r^2 + \omega^2[m_1m_2\omega^2 - k(m_1 + m_2)](S_1S_2)^{-1} = 0. \tag{10}$$

Since the discriminant of this biquadratic algebraic equation is positive,

$$W = [(m_1\omega^2 - k)S_1^{-1} - (m_2\omega^2 - k)S_2^{-1}]^2 + 4k^2(S_1S_2)^{-1} > 0,$$

there are two different real roots

$$r_{1,2}^2 = 0{,}5\{[(m_1\omega^2 - k)S_1^{-1} + (m_2\omega^2 - k)S_2^{-1}] \pm [(m_1\omega^2 - k)S_1^{-1} + (m_2\omega^2 - k)S_2^{-1}]^2 - 4\omega^2[m_1m_2\omega^2 - k(m_1 + m_2)](S_1S_2)^{-1})^{1/2}\}. \tag{11}$$

The analysis of these roots shows that solutions are possible for three cases. It can be proved that the root r_1^2 is always positive, while r_2^2 can be either positive, negative or equal to zero depending on the values of the parameters characterizing the vibrating system.

The following three cases are now considered:

Case 1: $r_1^2 > 0$ and $r_2^2 > 0$, if $\omega^2 > \omega_0^2$. The roots r_1^2 and r_2^2 are both positive, when

$$\omega^2 > \omega_0^2 = k(m_1^{-1} + m_2^{-1}) = K(M_1^{-1} + M_2^{-1}), \tag{12}$$

$$K = kl, \qquad M_i = m_il, \quad i = 1,2.$$

The frequency ω_0 denotes the natural frequency of a two-degree-of-freedom discrete system which consists of two rigid solids modelling the rigid strings joined by an elastic element.

This case is now considered in detail. The characteristic equation (10) has the following four real roots

$$r_s = +k_1, -k_1, +k_2, -k_2,$$

where

$$k_{1,2} = \{0{\cdot}5[(m_1\omega^2 - k)S_1^{-1} + (m_2\omega^2 - k)S_2^{-1}] \pm 0{\cdot}5([(m_1\omega^2 - k)S_1^{-1} + (m_2\omega^2 - k)S_2^{-1}]^2 - 4\omega^2[m_1 m_2\omega^2 - k(m_1 + m_2)](S_1S_2)^{-1})^{1/2}\}^{1/2}. \quad (13)$$

The general solutions (8) of equations (7) may now be written in the form

$$X_1(x) = \sum_{s=1}^{4} C_s e^{ir_s x} = \sum_{i=1}^{2} [A_i \sin(k_i x) + B_i \cos(k_i x)],$$
$$X_2(x) = \sum_{s=1}^{4} D_s e^{ir_s x} = \sum_{i=1}^{2} [C_i \sin(k_i x) + D_i \cos(k_i x)]. \quad (14)$$

The unknown constants A_i, B_i, C_i, D_i satisfy the relations resulting from equations (7),

$$C_i = a_i A_i, \qquad D_i = a_i B_i, \qquad i = 1{,}2,$$

where

$$a_i = (S_1 k_i^2 + k - m_1\omega^2)k^{-1} = k(S_2 k_i^2 + k - m_2\omega^2)^{-1}. \quad (15)$$

Then the general mode shapes of vibration (8) are as follows:

$$X_1(x) = \sum_{i=1}^{2} [A_i \sin(k_i x) + B_i \cos(k_i x)],$$
$$X_2(x) = \sum_{i=1}^{2} [A_i \sin(k_i x) + B_i \cos(k_i x)]a_i. \quad (16)$$

Finally, the free vibrations (5) of a double-string system are described by the expressions

$$w_1(x, t) = X_1(x)T(t) = [A \sin(\omega t) + B \cos(\omega t)] \sum_{i=1}^{2} [A_i \sin(k_i x) + B_i \cos(k_i x)],$$
$$w_2(x, t) = X_2(x)T(t) = [A \sin(\omega t) + B \cos(\omega t)] \sum_{i=1}^{2} [A_i \sin(k_i x) + B_i \cos(k_i x)]a_i. \quad (17)$$

The unknown constants A_i, B_i can be determined by solving the boundary value problem. Substituting the mode shape functions $X_1(x)$ and $X_2(x)$ (16) into the transformed boundary conditions (2)

$$X_1(0) = X_2(0) = X_1(l) = X_2(l) = 0$$

gives a set of four homogeneous algebraic equations for the unknown constants. For the existence of its non-trivial solutions the cardinal determinant of the coefficient matrix of equations must vanish. This necessary condition leads to the following characteristic equation:

$$\sin(k_i l) = 0, \quad i = 1, 2. \quad (18)$$

It can also be shown that $B_i = 0$. From the above relation, the unknown eigenvalues k_i can be calculated as

$$k_i = k_{in} = k_n = l^{-1} n\pi, \quad n = 1, 2, 3, \ldots . \tag{19}$$

The frequency equation of the vibration problem considered is obtained by transforming the relationship (13) and taking into account (19)

$$\omega^4 - [(S_1 k_n^2 + k) m_1^{-1} + (S_2 k_n^2 + k) m_2^{-1}]\omega^2 + k_n^2 [S_1 S_2 k_n^2 + k(S_1 + S_2)](m_1 m_2)^{-1} = 0. \tag{20}$$

The natural frequencies of a two-string system are calculated from the formula

$$\omega_{1,2n}^2 = 0{\cdot}5\{[(S_1 k_n^2 + k) m_1^{-1} + (S_2 k_n^2 + k) m_2^{-1}] \mp ([(S_1 k_n^2 + k) m_1^{-1} + (S_2 k_n^2 + k) m_2^{-1}]^2 - 4k_n^2 [S_1 S_2 k_n^2 + k(S_1 + S_2)](m_1 m_2)^{-1})^{1/2}\}, \tag{21}$$

$$\omega_{1n} < \omega_{2n}.$$

One can now formulate the time functions (6) and the natural mode shapes (16) corresponding to the natural frequencies ω_{in}

$$X_{1in}(x) = X_n(x) = \sin(k_n x), \qquad X_{2in}(x) = a_{in} X_n(x) = a_{in} \sin(k_n x), \tag{22}$$

$$T_{in}(t) = A_{in} \sin(\omega_{in} t) + B_{in} \cos(\omega_{in} t), \tag{23}$$

where

$$a_{in} = (S_1 k_n^2 + k - m_1 \omega_{in}^2) k^{-1} = k(S_2 k_n^2 + k - m_2 \omega_{in}^2)^{-1} = M_1(\omega_{11n}^2 - \omega_{in}^2) K^{-1} = K[M_2(\omega_{22n}^2 - \omega_{in}^2)]^{-1}, \quad i = 1, 2, \quad n = 1, 2, 3, \ldots, \tag{24}$$

$$K = kl, \qquad k_n = l^{-1} n\pi, \qquad M_i = m_i l = \rho_i F_i l, \qquad X_n(x) = \sin(k_n x),$$

$$\omega_{iin}^2 = (S_i k_n^2 + k) m_i^{-1} = (l S_i k_n^2 + K) M_i^{-1}, \qquad \omega_{12}^4 = k^2 (m_1 m_2)^{-1} = K^2 (M_1 M_2)^{-1}.$$

It is important to note that the coefficients a_{in} (24) are as follows:

$$a_{1,2n} = 0{\cdot}5 k^{-1} m_1 \{(\omega_{11n}^2 - \omega_{22n}^2) \pm [(\omega_{11n}^2 - \omega_{22n}^2)^2 + 4\omega_{12}^4]^{1/2}\},$$

$$a_{1n} > 0, \quad a_{2n} < 0, \quad a_{1n} a_{2n} = -m_1 m_2^{-1} = -M_1 M_2^{-1}.$$

Finally, the free vibrations of the system considered are described by the following formulae:

$$w_1(x, t) = \sum_{(i,n)} X_{1in}(x) T_{in}(t) = \sum_{n=1}^{\infty} X_n(x) \sum_{i=1}^{2} T_{in}(t) = \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} [A_{in} \sin(\omega_{in} t) + B_{in} \cos(\omega_{in} t)], \tag{25}$$

$$w_2(x,t) = \sum_{(i,n)} X_{2in}(x) T_{in}(t) = \sum_{n=1}^{\infty} X_n(x) \sum_{i=1}^{2} a_{in} T_{in}(t) = \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} [A_{in} \sin(\omega_{in} t) + B_{in} \cos(\omega_{in} t)] a_{in}.$$

Case 2: $r_1^2 > 0$ and $r_2^2 < 0$, if $\omega^2 < \omega_0^2$. The root r_1^2 is positive and r_2^2 is negative, when $\omega^2 < \omega_0^2$.

For this case the general mode shapes of vibration (8) are as follows:

$$X_1(x) = \sum_{i=1}^{2} X_{1i}(x) = A_1 \sin(k_1 x) + B_1 \cos(k_1 x) + A_2 \sinh(k_2 x) + B_2 \cosh(k_2 x),$$

$$X_2(x) = \sum_{i=1}^{2} a_i X_{1i}(x) = [A_1 \sin(k_1 x) + B_1 \cos(k_1 x)]a_1 + [A_2 \sinh(k_2 x) + B_2 \cosh(k_2 x)]a_2,$$

where

$$a_1 = (S_1 k_1^2 + k - m_1\omega^2)k^{-1} = k(S_2 k_1^2 + k - m_2\omega^2)^{-1},$$

$$a_2 = (k - S_1 k_2^2 - m_1\omega^2)k^{-1} = k(k - S_2 k_2^2 - m_2\omega^2)^{-1},$$

$$k_{1,2} = \{ \pm 0{\cdot}5[(m_1\omega^2 - k)S_1^{-1} + (m_2\omega^2 - k)S_2^{-1}] + 0{\cdot}5([(m_1\omega^2 - k)S_1^{-1}$$
$$+ (m_2\omega^2 - k)S_2^{-1}]^2 - 4\omega^2[m_1 m_2 \omega^2 - k(m_1 + m_2)](S_1 S_2)^{-1})^{1/2}\}^{1/2},$$

$$X_{11}(x) = A_1 \sin(k_1 x) + B_1 \cos(k_1 x), \qquad X_{12}(x) = A_2 \sinh(k_2 x) + B_2 \cosh(k_2 x).$$

Solving the boundary value problem gives the following values of constants:

$$B_1 = A_2 = B_2 = 0$$

and the characteristic equation in the form $\sin(k_1 l) = 0$. Thus, the unknown eigenvalue k_1 can be calculated as (19)

$$k_1 = k_{1n} = k_n = l^{-1} n\pi, \quad n = 1,2,3, \ldots.$$

Then the natural mode shapes are received in the same form (22) as for the case 1. The eigenfrequencies are computed from the formula (21), and finally the free vibrations are expressed by identical relations as equations (25).

Case 3: $r_1^2 > 0$ and $r_2^2 = 0$, if $\omega^2 = \omega_0^2$. The root r_1^2 is positive and r_2^2 is equal to zero, when $\omega^2 = \omega_0^2$.

For this case the general mode shapes of vibration (8) are as follows:

$$X_1(x) = \sum_{i=1}^{2} X_{1i}(x) = A_1 \sin(k_1 x) + B_1 \cos(k_1 x) + A_2 x + B_2,$$

$$X_2(x) = \sum_{i=1}^{2} a_i X_{1i}(x) = [A_1 \sin(k_1 x) + B_1 \cos(k_1 x)]a_1 + [A_2 x + B_2]a_2,$$

where

$$a_1 = m_2 S_1 (m_1 S_2)^{-1}, \qquad a_2 = -m_1 m_2^{-1}, \qquad k_1^2 = k[m_1(m_2 S_1)^{-1} + m_2(m_1 S_2)^{-1}],$$

$$X_{11}(x) = A_1 \sin(k_1 x) + B_1 \cos(k_1 x), \qquad X_{12}(x) = A_2 x + B_2.$$

Solving the boundary value problem gives $A_1 = B_1 = A_2 = B_2 = 0$. This means that free vibrations with the natural frequency $\omega = \omega_0$ are impossible for the string system. The frequency ω_0 denotes the natural frequency of a two-degree-of-freedom discrete system consisting of two rigid solids (modelling the rigid strings) joined by an elastic element. The boundary conditions make it impossible to execute the vibration motion of an elastically connected two rigid string system. Finally, it is seen that the cases for which the roots $r_2^2 \leqslant 0$ are of no interest in the free vibration analysis of the system considered.

The boundary conditions imposed mean that the solutions to the problem are based on the positive roots (11) only ($r_1^2 > 0$ and $r_2^2 > 0$) and it can be shown that the double-string system performs the free harmonic vibrations described by equations (25).

It should be noted that the homogeneous partial differential equations (4) with the particular boundary conditions (2) can be solved by using less general mathematical procedures (applying the Fourier-series method) assuming the solutions to be in the form:

$$w_i(x, t) = \sum_{n=1}^{\infty} X_n(x) S_{in}(t) = \sum_{n=1}^{\infty} \sin(k_n x) S_{in}(t), \quad i = 1,2,$$

where $S_{in}(t)$ is the unknown time function and $X_n(x)$ is the known mode shape function for a single string. Solving this problem gives the following time functions:

$$S_{1n}(t) = \sum_{i=1}^{2} T_{in}(t) = \sum_{i=1}^{2} [A_{in} \sin(\omega_{in} t) + B_{in} \cos(\omega_{in} t)],$$

$$S_{2n}(t) = \sum_{i=1}^{2} a_{in} T_{in}(t) = \sum_{i=1}^{2} [A_{in} \sin(\omega_{in} t) + B_{in} \cos(\omega_{in} t)] a_{in}.$$

These solutions are identical to equations (25).

Analyzing the solutions (25) which express the free vibrations of the system considered allows an important conclusion to be drawn. An elastically connected double-string complex system executes two kinds of vibrating motions: synchronous vibrations ($a_{1n} > 0$) with lower frequencies ω_{1n} and asynchronous vibrations ($a_{2n} < 0$) with higher frequencies ω_{2n} ($\omega_{1n} < \omega_{2n}$). The corresponding general mode shapes of vibrations of the system are shown in Figure 2.

The unknown constants A_{in} and B_{in} are determined from the assumed initial conditions (3). Solving the initial-value problem the orthogonality condition of mode shapes of vibration is applied. The orthogonality condition is derived using the equations (7) rewritten in the following form:

$$S_1 X''_{1in} + (m_1 \omega_{in}^2 - k) X_{1in} + k X_{2in} = 0, \qquad S_2 X''_{2in} + (m_2 \omega_{in}^2 - k) X_{2in} + k X_{1in} = 0.$$

The above equations must be satisfied by the natural shape functions of the strings. Substituting the expressions (22) into these equations yields

$$S_1 X''_n + [m_1 \omega_{in}^2 + (a_{in} - 1) k] X_n = 0, \qquad S_2 X''_n + [m_2 \omega_{in}^2 + (a_{in}^{-1} - 1) k] X_n = 0.$$

Together with the relations (24) the above equations are transformed into the simple equation used for a single string

$$X''_n + k_n^2 X_n = 0. \tag{26}$$

The eigenfunction $X_n(x) = \sin(k_n x)$ solves this equation and satisfies the classical orthogonality condition as follows

$$\int_0^l X_m X_n \, \mathrm{d}x = \int_0^l \sin(k_m x) \sin(k_n x) \, \mathrm{d}x = c \delta_{mn}, \tag{27}$$

$$c = c_n^2 = \int_0^l X_n^2 \, \mathrm{d}x = \int_0^l \sin^2(k_n x) \, \mathrm{d}x = 0{\cdot}5 l,$$

where δ_{mn} is the Kronecker delta function: $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{mn} = 1$ for $m = n$.

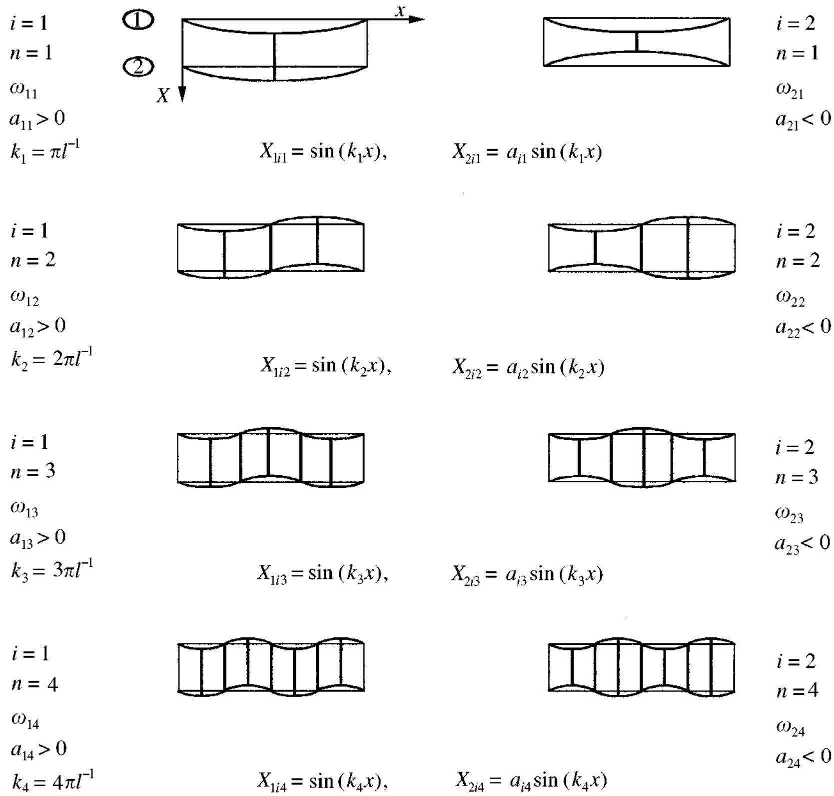

Figure 2. The general mode shapes of vibrations of an elastically connected double-string complex system corresponding to the first four pairs of the natural frequencies.

Substitution of the solutions (25) into the initial conditions (3) gives

$$w_{10} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} B_{in}, \qquad v_{10} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} \omega_{in} A_{in},$$

$$w_{20} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} a_{in} B_{in}, \qquad v_{20} = \sum_{n=1}^{\infty} X_n \sum_{i=1}^{2} a_{in} \omega_{in} A_{in}.$$

Multiplying the above relations by the eigenfunction X_m, then integrating them with respect to x from 0 to l and using the orthogonality condition (27) produces

$$\int_0^l w_{10} X_n \,\mathrm{d}x = c \sum_{i=1}^{2} B_{in}, \qquad \int_0^l v_{10} X_n \,\mathrm{d}x = c \sum_{i=1}^{2} \omega_{in} A_{in},$$

$$\int_0^l w_{20} X_n \,\mathrm{d}x = c \sum_{i=1}^{2} a_{in} B_{in}, \qquad \int_0^l v_{20} X_n \,\mathrm{d}x = c \sum_{i=1}^{2} a_{in} \omega_{in} A_{in}.$$

Solving these equations gives the following formulae which make it possible to calculate the unknown constants:

$$A_{1n} = (c_{1n}\omega_{1n})^{-1} \int_0^l (a_{2n} v_{10} - v_{20}) \sin(k_n x) \,\mathrm{d}x,$$

$$A_{2n} = (c_{2n}\omega_{2n})^{-1} \int_0^l (a_{1n}v_{10} - v_{20}) \sin(k_n x)\, \mathrm{d}x,$$

$$B_{1n} = c_{1n}^{-1} \int_0^l (a_{2n}w_{10} - w_{20}) \sin(k_n x)\, \mathrm{d}x, \tag{28}$$

$$B_{2n} = c_{2n}^{-1} \int_0^l (a_{1n}w_{10} - w_{20}) \sin(k_n x)\, \mathrm{d}x,$$

where

$$c_{1n} = -c_{2n} = (a_{2n} - a_{1n})c = 0{\cdot}5l(a_{2n} - a_{1n}).$$

The free vibration problem of a double-string system is finally solved.

4. CONCLUSIONS

The transverse vibration theory of an elastically connected double-string complex system has been derived. The free vibrations of two parallel strings joined by a Winkler elastic element are considered. The motion of the system is described by a non-homogeneous conjugate set of two partial differential equations. The solutions of the free vibrations are formulated by the Bernoulli–Fourier method. By solving the boundary value and initial-value problems, the natural frequencies and natural mode shapes of vibration are found. The free vibrations of a double-string system are realized by two kinds of motions: synchronous vibrations ($a_{1n} > 0$) with lower frequencies ω_{1n} and asynchronous vibrations ($a_{2n} < 0$) with higher frequencies ω_{2n} ($\omega_{1n} < \omega_{2n}$).

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